Registration number

B.E./B.Tech (Full time) Degree End Semester Examination, April / May 2024

Fourth Semester

MA7355 PROBABILITY AND QUEUEING THEORY

(Regulation 2015/2019)

(Use of Statistical Tables and Calculators are permitted)

Max Marks: 100 ANSWER ALL QUESTIONS Time: 3 Hours

PART A (10 x 2 = 20)

1. A continuous random variable has the probability density function given by $f(x) = k(x-1)^3, 1 \le x \le 3$. Find the value of k.

2. Let X be a random variable which follows uniform distribution in $(2, 9)$, find $P(X > 4)$.

3. Find the value of k, if the joint density function of (X, Y) is given by

$$f(x,y) = \begin{cases} k(1-x)(1-y), & 0 < x < 4, 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

4. The equations of regression lines obtained in a correlation analysis are $3X + 12Y = 19, \quad 3Y + 9X = 46$. Obtain the means of the distribution.

5. Define strict sense and wide sense stationary process.

6. Find the mean and second moment of a stationary random process whose autocorrelation function is given by $R_{XX}(\tau) = 18 + \dfrac{2}{6+\tau^2}$.

7. Define Kendall's notation.

8. Present the formula for the steady state probabilities of the finite source model.

9. Consider a system with two infinite queues in series, where each of the two service facilities has a single server. All service times are independent and have an exponential distribution, with mean of 3 minutes at facility 1 and 4 minutes at facility 2. Facility 1 has a Poisson input process with a mean rate of 10 per hour. Find the expected total number of customers in the system.

10. Distinguish between open and closed queueing network.

PART B (5 x 16 = 80)

11. (a) i. Derive the Pollaczek-Khintchine formula for the average number in the system in an $M/G/1$ queueing model. (10 Marks)

ii. An average of 120 students arrive each hour (inter-arrival times are exponential) at the controller office to get their hall tickets. To complete this process, a candidate must pass through three counters. Each counter consists of a single server. Service times at each counter are exponential with the following mean times: counter 1, 20 seconds; counter 2, 15 seconds and counter 3, 12 seconds. On an average, how many students will be present in the controller's office. (6 Marks)

12. (a) i. Find the mean, variance and moment generating function of a Poisson random variable. (8 Marks)

ii. The mileage which the car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40,000km. Find the probability that one of these tyres will last (A) atleast 20,000 km and (B) atmost 30,000 km. (8 Marks)

[OR]

(b) i. Find the mean, variance and moment generating function of an exponential random variable. (8 Marks)

ii. In a class of 50, the average mark of students in a subject is 48 and the standard deviation is 24. Find the number of students who get (A) above 50 (B) between 35 to 50. (8 Marks)

13. (a) i. X and Y are two continuous random variables whose joint probability density function is given by

$$f_{XY}(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4} & 0 \le x < 2, 0 \le y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? (8 Marks)

ii. Let X and Y be random variables having joint probability density function

$$f(x,y) = \begin{cases} \dfrac{6-x-y}{8}, & 0 \le x \le 2, 2 \le y \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional density functions of X on Y and that of Y on X.

(8 Marks)

[OR]

(b) The joint PDF of two random variables X and Y is given by

$$f(x,y) = \frac{3}{2}(x^2+y^2), \quad 0 \le x \le 1, 0 \le y \le 1.$$

Find the lines of regression of X on Y and Y on X.

(16 Marks)

14. (a) i. If $\{N_1(t)\}$ and $\{N_2(t)\}$ are two independent Poisson process with parameter $\lambda_1 t$ and $\lambda_2 t$ respectively, show that

$$P(N_1(t)=k/N_1(t)+N_2(t)=n) = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}.$$

(8 Marks)

ii. Let $\{X_n\}$ be a Markov chain with state space $\{0,1,2\}$ with initial probability vector $p^{(0)} = (0.7, 0.2, 0.1)$ and the one step transition probability matrix

$$P = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

Compute (A) $P(X_2 = 3)$ (B) $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2)$.

(8 Marks)

[OR]

(b) i. Given $X(t) = A\cos(\omega t + \theta)$, where A and ω are constant and θ is uniformly distributed in $(0, 2\pi)$, is $\{X(t)\}$ a WSS process.

(8 Marks)

ii. Find the limiting probabilities associated with the transition probability matrix of a Markov chain given by $P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$.

(8 Marks)

15. (a) i. Derive the steady state probabilities of an $(M/M/1):(FCFS/N/\infty)$ queuing model and obtain a closed form expression for the average number of customer in the system. (8 Marks)

ii. The library has two counters at present and borrowers arrive according to Poisson distribution with arrival rate of 1 every 6 minutes and the service time follows exponential distribution with a mean of 10 minutes. Find the average waiting time in the queue for the borrowers.

(8 Marks)

[OR]

(b) i. Derive the steady state probabilities of an $(M/M/c):(FCFS/\infty/\infty)$ model and obtain a closed form expression for the average number of customers waiting in the queue.

(8 Marks)

ii. A car park has 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has exponential distribution with mean of 2 minutes. How many cars are in the car park on an average and what is the probability of the newly arriving customer finding the car park full and leaving to park his car elsewhere. (8 Marks)